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Fibred Links and a Construction of Real Singularities Via Complex Geometry

Jose Seade[1]

Abstract. This note gives a method for constructing real analytic maps from $\mathbb{R}^{2n}$ into $\mathbb{R}^2$, with an isolated critical point at $0 \in \mathbb{R}^{2n}$, for all $n > 1$. This provides infinite families of real singularities which fiber "a la Milnor".

0. Introduction

In this note I give a method for constructing real analytic maps from $\mathbb{R}^{2n}$ into $\mathbb{R}^2$, with an isolated critical point at $0 \in \mathbb{R}^{2n}$, for all $n > 1$. These maps have a very rich geometry, which is a reminiscent of the geometry of complex singularities [**Mi**], and in some ways it is even richer. This provides infinite families of real singularities which fibre "a la Milnor" [**Mi, p.100**]. The construction arises from the work of Arnold [**Ar**], Camacho-Kuiper-Palis [**CKP**] and Gomez Mont-Verjovsky and myself [**GSV, Se**], by studying holomorphic vector fields from the differentiable point of view.

The construction is in fact simple: Let $\chi(\mathbb{C}^n, 0)$ denote the space of all germs of continuous vector fields at $0 \in \mathbb{C}^n$, and let F, X be elements in $\chi(\mathbb{C}^n, 0)$. One has a continuous map,

$$\phi_{F,X}: \mathbb{C}^n \cong \mathbb{R}^{2n} \to \mathbb{C} \cong \mathbb{R}^2,$$

defined by $\psi_{F,X}(z) = \langle F(z), X(z) \rangle$, where

$$\langle F(z), X(z) \rangle = \sum_{i=1}^{n} F_i(z) \cdot \overline{X}_i(z),$$

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is the usual hermitian product. These are the maps to which the title of this article refers. We note that if F and X are both differentiable of class C^r, then $\psi_{F,X}$ is of class C^r, if they are both real analytic, $\psi_{F,X}$ is real analytic, but if F and X are complex analytic, then $\psi_{F,X}$ is not complex analytic, unless X or F are constant. So the singularities we obtain are truly real singularities, though we shall be considering holomorphic vector fields.

As an example, let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the Pham-Brieskorn polynomial $f(z) = z_1^p + z_2^q$, with p, $q > 2$, let

$$F = \left(\frac{\partial f}{\partial z_2}, -\frac{\partial f}{\partial z_1}\right),$$

a holomorphic vector field whose solutions are the fibres of f, and take X to be a constant vector field $X = (a_1, a_2)$. Then $\psi_{F,X}$ is a holomorphic function with an isolated critical point at 0 and $M = \psi_{F,X}^{-1}(0)$ is the polar curve determined by a linear form (studied by Teissier, Lê and others, still unpublished). Now take F to be the linear vector field in $\mathbb{C}^3$ which is in the Siegel domain with generic eigenvalues [**CKP**], for instance $F = (z_1, iz_2, (-1-i)\cdot z_3)$, and take X to be any linear vector field $X(z) = (z_1, z_2, z_3)$; then $M = \psi_{F,X}^{-1}(0)$ is the cone over the 3-torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ [**CKP, Lm**]. Hence $M = \psi_{F,X}^{-1}(0)$ is not a complex singularity, because the 3-torus is not the link of any complex surface singularity [**Su**]. Though M is not a complex singularity, $M - \{0\}$ is canonically a complex manifold with a holomorphic C^*-action with compact quotient $M/\mathbb{C}^*$, which is not a projective manifold [**LV**]; M is the space of Siegel leaves of F.

Although some things can be done in general, for F and X continuous vector fields, I prefer to restrict myself to what I think is the ideal environment: F holomorphic and X the gradient vector field of a real analytic function f. Then the variety M of $\psi_{F,X}$ is the polar variety of the foliation $\mathcal{F}$ of F and the foliation given by the level surfaces of f, i.e. M is the set of points where the two foliations are tangent.

In §1 below we look at the geometry of the functions $\psi_{F,X}$ in general. In §2 we look at the varieties defined by these maps, the polar varieties (or contact sets), and in §3 we show that this construction produces singularities which fibre a la Milnor [**Mi**], i.e. they give rise to locally trivial fibre bundles of the type,

$$\phi: \mathbb{S}_\epsilon - M \to \mathbb{S}^1,$$
$$z \mapsto \frac{f(z)}{\|f(z)\|},$$

where f is now the map $\psi_{F,X}$. The existence of this type of examples was asked by Milnor on page 100 of his book. This question was answered positively by Looijenga in [**Lo**], by proving that for every $n > 1$ there exists a real polynomial map $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^2, 0)$ defining a fibration of Milnor's type. However, Looijenga's proof is not constructive in the sense that it does not give explicit polynomials for which one has such fibrations. (See also [**Pe**].). Our construction gives infinite families of such examples. In §4 we study in detail one such family. We show that in these cases there are surprising analogies with [**Mi**] regarding the topology of the fibers. I do not know if this is a coincidence or if this is a special case of some general theorem.

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1. The Geometry of $\psi_{F,X}$

Let F and X be elements in $\chi(\mathbb{C}^n, 0)$, with F holomorphic and X being the gradient vector field of some real analytic function

$$f: \mathbb{C}^n \cong \mathbb{R}^{2n} \to \mathbb{R},$$

with an isolated critical point at 0. The level surfaces of f, $\mathcal{V}_t = f^{-1}(t)$, are normal to X and define a (real) codimension 1 foliation $\mathcal{S}$ of $\mathbb{C}^n$, singular at 0. Let $\mathcal{F}$ be the holomorphic foliation by complex curves defined by F.

1.1 Definition (c.f. [**Th**]). The polar variety of f and F is the set M of points in $\mathbb{C}^n$ where the foliations $\mathcal{F}$ and $\mathcal{S}$ are tangent.

It is clear that one has,

$$M = \{z \in \mathbb{C}^n \mid \langle F(z), X(z) \rangle = 0\},$$

so it is a real analytic space defined by the equations:

$$\operatorname{Re}\langle F(z), X(z) \rangle = 0,$$
$$\operatorname{Im}\langle F(z), X(z) \rangle = 0.$$

Away from M these two foliations meet transversely, defining a foliation Γ by real curves.

1.2 Lemma. *The curves of* Γ *are the integrals of the real analytic vector field,*

$$\tau(z) = (i\overline{\langle F(z), X(z) \rangle}) \cdot F(z),$$

whose zero locus is M.

Proof. It is clear that $\tau(z)$ is always tangent to $\mathcal{F}$, because at each point $z \in \mathbb{C}^n$, $\tau(z)$ is $F(z)$ multiplied by a complex number. So we must prove that $\tau(z)$ is normal to $X(z)$. One has,

$$\begin{aligned}\langle \tau(z), X(z) \rangle &= \langle (i\overline{\langle F(z), X(z) \rangle}) \cdot F(z), X(z) \rangle \\ &= (i\overline{\langle F(z), X(z) \rangle}) \cdot \langle F(z), X(z) \rangle \\ &= i\|\langle F(z), X(z) \rangle\|^2.\end{aligned}$$

Hence $\operatorname{Re}\langle \tau(z), X(z) \rangle = 0$, so $\tau(z)$ is normal to $X(z)$, because the real part of the hermitian product is the usual inner product in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. $\square$

This vector field gives the dynamical behaviour of F in the direction determined by the level surfaces of f. For instance, if $X(z) = z$ and F is a linear vector field in the Poincaré domain with generic eigenvalues, then τ is Morse-Smale [**Gu**], which is used in [**CKP**] to prove that the linear vector fields in the domain of Poincaré are structurally stable. If we multiply τ by the complex number i, we get another vector field, which commutes with τ and gives the dynamical behavior of F in the direction transversal to the level surfaces of f.

The following proposition gives a geometric interpretation of the map $z \to i\overline{\langle F(z), X(z)\rangle}$, which is essentially the map mentioned in the introduction of this article.

1.3 Proposition. *For each $z \in \mathbb{C}^n - M$, the argument of the complex number $i\overline{\langle F(z), X(z)\rangle}$ is the angle by which we must rotate the vector $F(z)$ in its complex line, to make it tangent to the level surface of f at z.*

Define a function $\phi = \phi_{F,X}: \mathbb{C}^n - M \to \mathbb{S}^1 \subset \mathbb{C}$ by,

$$\phi(z) = \frac{i\overline{\langle F(z), X(z)\rangle}}{\left\| i\overline{\langle F(z), X(z)\rangle} \right\|},$$

and set $E_\theta = \phi^{-1}(e^{i\theta})$. For each $\theta \in [0, \pi) \subset \mathbb{R}$, we define a map,

$$\psi_\theta: \mathbb{C}^n \to \mathbb{R},$$

by $\psi_\theta(z) = \mathrm{Re}\langle e^{i\theta} F(z), X(z)\rangle$. We set,

$$M_\theta = \psi_\theta^{-1}(0) = \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle e^{i\theta} F(z), X(z)\rangle = 0\}.$$

M_0 is the set of points where $F(z)$ is orthogonal over $\mathbb{R}$ to $X(z)$; $M_{\pi/2}$ is the set of points where $iF(z)$ is orthogonal over $\mathbb{R}$ to $X(z)$, and so on.

One has the following decomposition theorem:

1.4 Theorem.

i) $C^n = \cup M_\theta$, $\theta \in [0, \pi)$.

ii) $M = \cap M_\theta$, $\theta \in [0, \pi)$.

iii) $M_\theta = E_\theta \cup M \cup E_{\theta+\pi}$, *for each* θ.

Proof. Statement (ii) is clear and statement (i) follows from statement (iii), so we prove (iii). It is also clear that $M \subset M_\theta$. Let us prove that $E_\theta \subset M_\theta$. If $z \in E_\theta$ then,

$$\frac{e^{-i\theta} i\overline{\langle F(z), X(z)\rangle}}{\left\| i\overline{\langle F(z), X(z)\rangle} \right\|} = 1,$$

thus,

$$e^{-i\theta} i\overline{\langle F(z), X(z)\rangle} = i\overline{\langle e^{i\theta} F(z), X(z)\rangle},$$

is a real number. Hence $\operatorname{Re}\langle e^{i\theta}F(z), X(z)\rangle = 0$, so z is in M_θ. The prove that $E_{\theta+\pi} \subset M_\theta$ is similar, so we leave it. Let us prove,

$$M_\theta \subset E_\theta \cup M \cup E_{\theta+\pi}.$$

If $z \in M_\theta$ then $i\langle e^{i\theta}F(z), X(z)\rangle \in \mathbb{R}$. If $\langle F(z), X(z)\rangle \neq 0$, this implies,

$$\frac{e^{i\theta}\overline{i\langle F(z), X(z)\rangle}}{\left\|\overline{i\langle F(z), X(z)\rangle}\right\|} = \pm 1,$$

and that z is in E_θ or in $E_{\theta+\pi}$, depending on the sign in the right hand side. □

2. The Polar Varieties

In this section we give some general properties of the polar varieties arising by the construction of §1 above. We start with a few examples, giving an insight of the difficulty for understanding these singularities:

2.1 Examples. a. Let $F = (\lambda_1, z_1, \lambda_2, z_2, \lambda_3, z_3)$ in $\mathbb{C}^3$ and $X = (z_1, z_2, z_3)$, so $M = \{z \in \mathbb{C}^3 \mid \sum_{i=1}^{3} \lambda_1 \|z_1\|^2 = 0\}$.

i) If $\operatorname{Re}\lambda_i > 0$ for $i = 1, 2, 3$, then $M = \{0\}$.

ii) If $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = i$, then M is given by the equations:

$$\|z_1\| = \|z_2\|, z_3 = 0;$$

M is the cone over the 2-torus, it has real codimension 3.

iii) If $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -1$, then M is given by the equation:

$$\|z_1\|^2 + \|z_2\|^2 = \|z_3\|^2,$$

so it is a codimension 1 real quadric.

iv) If $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = -1 + i$, then M is given by:

$$\|z_1\| = \|z_2\| = \|z_3\|,$$

which is the cone over the 3-torus. It has codimension 2.

b. Our next example is due to T. Ito [**It, p. 186**]: Let $X = (z_1, z_2)$ and let

$$F = (2z_1 + (1+i)z_2^2, z_2).$$

Then M has two connected components: One component consists of $0 \in \mathbb{C}^2$, which is an isolated point of M, and the other component M_1 has a circle $\mathbb{S}^1$ as singular set $\sum$; if we remove $\sum$ from M_1 what we get is the disjoint union of two cylinders homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, which are transversal to all the spheres around 0.

We see in these examples that the behaviour of the corresponding polar variety changes drastically in each case. Still, one has the following theorem, which is an easy extension of Theorem 2.3 in [**GSV**], so we state it without proof.

2.2 Theorem. *M is a real geometric complete intersection in $\mathbb{C}^n$, of real codimension 2, with a unique singular point at $0 \in \mathbb{C}^n$ if and only if all the contacts of $\mathcal{F}$ and $\mathcal{S}$ are generic, except at 0. In this case $M^* = M - \{0\}$ is canonically a complex $(n-1)$-manifold.*

A contact of two foliations is a point where they are tangent, i.e. a point in M. We refer to the article of R. Thom [**Th**] for a discussion of "generic contacts". The idea is simple: X is a gradient vector field of some function f. By restricting f to the leaves of $\mathcal{F}$, one gets a vector field $X_{\mathcal{F}}$ whose solutions are contained in the leaves of $\mathcal{F}$. The zeros of $X_{\mathcal{F}}$ are the points in M, and the contact is generic when the corresponding zero of $X_{\mathcal{F}}$ is non-degenerate. Hence, these are contacts which are either non-degenerate local minimal points in $\mathcal{F}$, or local maximal points or saddles. It follows (c.f. [**GSV, Theorem 2.3**]) that if M is a complete intersection with an isolated singularity at 0, and if it has a point $x \in M - \{0\}$ which is either a minimal point, a saddle or a maximal point, then the connected component of $M - \{0\}$ that contains x consists of contacts of the same type.

2.3 Example. Let F be a linear vector field in the Siegel domain with generic eigenvalues (see [**CKP,Lm**]) and let $X(z) = z$. Then M is a complete intersection with a unique singular point at 0. It is the intersection of two real quadrics, and it is a cone with vertex at 0. The base of this cone is the intersection of M with the unit sphere, and it may have a very interesting topology, as it is shown in [**Wa,Lm,LV**], where these varieties are classified.

3. About Milnor's Fibration Theorem

J. Milnor in [**Mi**] proved the following fibration theorem:

Theorem . *Let* $f: (\mathcal{U} \subset \mathbb{R}^{n+k}, P) \to (\mathbb{R}^k, 0)$ *be a real analytic function with a critical point at* P*, such that for each point* $x \neq P$ *near* P*, the Jacobian matrix* $Df(x)$ *has rank* k*. Let* $\mathcal{V} = f^{-1}(0)$*, let* $\mathbb{S}_\epsilon$ *be a small sphere around* P *and let* $\mathcal{N} = \mathcal{N}(\mathcal{V})$ *be a tubular neighbourhood of* $\mathcal{V} \cap \mathbb{S}_\epsilon$ *in* $\mathbb{S}_\epsilon$*. Then one has a function,*

$$\phi: \mathbb{S}_\epsilon - \mathcal{N} \to \mathbb{S}^{k-1},$$

which is the projection map of a C^∞ *fibre bundle.*

We proved in [**Se**] that the method above produce infinite real singularities satisfying Milnor's hypothesis: whenever F is a vector field in $\mathbb{C}^n$ of the form,

$$F(z) = (k_1 z_{i_1}^{a_1}, \ldots, k_n z_{in}^{an}),$$

where the k_i's are all non-zero complex numbers and the a_i's are integers > 1, and if X is the radial vector field $X(z) = (z_1, \ldots, z_n)$, then the map $\psi_{F,X}$ satisfies Milnor's hypothesis.

We note that in all these cases $\psi_{F,X}$ satisfies Milnor's hypothesis, one has the fibre bundle,

$$\phi: \mathbb{C}^n - \mathcal{N}(M) \to \mathbb{S}^1 \subset \mathbb{C},$$

where

$$\phi = \frac{\psi_{F,X}}{\|\psi F, X\|},$$

and Theorem 1.4 above implies that each pair of antipodal fibres is glued together along M forming a real analytic space homeomorphic to,

$$M_0 = \{z \in \mathbb{C}^n \mid \mathrm{Re}\langle F(z), X(z)\rangle = 0\}.$$

This is necessarily a real hypersurface in $\mathbb{C}^n$ with an isolated singularity at 0, c.f. [**Pa**].

We now give more examples of vector fields F and X for which the corresponding function $\psi_{F,X}$ satisfies Milnor's hypothesis, so one has an associated fibre bundle. In these examples X is no longer the radial vector field.

3.1 Example. Let $f\colon \mathbb{C}^{2n} \to \mathbb{C}$ be the Pham-Brieskorn polynomial,

$$f(z) = z_1^{a_1} + \cdots + z_{2n}^{a_{2n}},$$

with $a_i > 2$ for all i. Set

$$f_i = \frac{\partial f}{\partial z_i}$$

and let F be any hamiltonian vector field of the form

$$(f_2, -f_1, \ldots, f_{2n}, -f_{2n-1}), (f_3, f_4, -f_1, -f_2, \ldots),$$

etc., whose solutions are contained in the fibres of F. Let

$$X = (a_1, \ldots, a_n)$$

constant. Then $\phi_{F,X}$ is a holomorphic function with an isolated critical point at 0. Hence the polar variety M is a complex hypersurface in $\mathbb{C}^{2n}$ with an isolated singularity at 0, so it has an associated Milnor fibration. It would be interesting to understand the relationship among these fibrations as we take different hamiltonian vector fields, and also their relationship with the original Milnor fibration of the function f. An extra bonus we have in these cases is that Theorem 1.4 above tells us that the double of the Milnor fibre is the intersection of the unit sphere $\mathbb{S}^{4n-1}$ with the real analytic variety,

$$M_0 = \{z \in \mathbb{C}^{2n} \mid \operatorname{Re} \sum_{i=1}^{2n} \overline{a}_i F_i(z) = 0\}.$$

3.2 Example. Let $F(z) = (z_1^{a_1}, \ldots, z_n^{a_n})$, and $X = (1, \ldots, l, z_{r+1}, \ldots, z_n)$. Then,

$$\psi_{F,X}(z) = z_1^{a_1} + \cdots + z_r^{a_r} + z_{r+1}^{a_{r+1}} \overline{z}_{r+1} + \cdots + z_n^{a_n} \overline{z}_n.$$

We write,

$$\psi_{F,X}(z) = \frac{1}{2}(\psi_{F,X}(z) + \overline{\psi}_{F,X}(z), \psi_{F,X}(z) - \overline{\psi}_{F,X}(z)),$$

and we consider its derivatives with respect to $z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n$. This is a $2 \times 2n$ matrix. The determinant of the first 2×2 minor is 0 if and only if,

$$-2a_1^2\|z_1\|^{2a_1-2} = 0,$$

i.e. if and only if $z_1 = 0$. The same statement holds for $z_2, \ldots, z_r$. Considering the minor given by the partial derivatives with respect to z_{r+1} and $\overline{z}_{r+1}$ we see that its determinant is 0 if and only if,

$$a_{r+1}^2\|z_{r+1}\|^{2(a_r+1)} = \|z_{r+1}\|^{2a_r+1},$$

and similarly for $a_{r+2}, \ldots, a_n$. This happens if and only if $z_i = 0$ for all z_i, $i > r$. Hence $\psi_{F,X}$ satisfies Milnor's hypothesis.

3.3 Example. The following is an example where the vector field X is not necessarily linear. Let $F = (z_1^p, z_2^q)$ and $X = (z_1^r, z_2^s)$, with p, q, r, $s > 0$ and $p \neq r$, and $q \neq s$.

The proof that these examples satisfy Milnor's hypothesis is exactly as in the example above, so we omit the details. One decomposes $\psi_{F,X}$ into its real part and its imaginary part, and considers the jacobian matrix taking derivatives with respect to the z_i's and the $\overline{z}_i$'s. Then one shows that there is always a 2×2 minor whose determinant is $\neq 0$, unless $z_i = 0$ for all i.

3.4 Example. As a last example, consider F to be a linear vector field, $F(z) = (\lambda_1 z_1, \ldots, \lambda_n z_n)$. It is easy to see that if we take X to be the radial vector field $X(z) = (z_1, \ldots, z_n)$, then the corresponding function $\phi_{F,X}$ never satisfies Milnor's hypothesis. The rank of its jacobian matrix drops at the axes. However, suppose $n = 2n$ is even, and take $X(z) = (z_2, -z_1, \ldots, z_{2r}, -z_{2r-1})$, then it is an exercise to show that $\psi_{F,X}$ satisfies Milnor's hypothesis if and only if $\|\lambda_1\| \neq \|\lambda_2\|$, $\|\lambda_3\| \neq \|\lambda_4\|$, and so on until $\|\lambda_{2r-1}\| \neq \|\lambda_{2r}\|$.

It is worth noting that one produces more examples by mixing the previous ones. Whenever (F_i, X_i), $i = 1, \ldots, r$, are pairs of vector fields in $\mathbb{C}^{n_i}$ for which the map ψ_{F_i,X_i} satisfies Milnor's hypothesis, the direct sum $(F_1 \oplus \cdots \oplus F_r, X_1 \oplus \cdots \oplus X_r)$ is a pair of vector fields in $\mathbb{C}^{n_1+\cdots+n_r}$

satisfying Milnor's hypothesis.

4. The Topology of the Fibres: An Example

Let $(F(z_1, z_2)$ be the vector field in $\mathbb{C}^2$ defined by,

$$F(z_1, z_2) = (z_2^k, -z_1^k), \quad k > 1.$$

Let M be the set of points $z = (z_1, z_2)$ in $\mathbb{C}^2$ where $F(z)$ is tangent to the spheres around 0. That is,

$$M = \{z \in \mathbb{C}^2 \mid \langle F(z), z\rangle = 0\},$$

where $\langle F(z), z\rangle = z_2^k \cdot \bar{z}_1 - z_1^k \cdot \bar{z}_2$ is the hermitian product.

We first state the results, the proofs come later.

4.1 Theorem. *M has real dimension 2, it is smooth away from $0 \in \mathbb{C}^2$, and it is embedded in $\mathbb{C}^2$ as the cone over a link $\mathbf{L}$ in $\mathbb{S}^3$ with $(k+3)$ components, each being a fibre of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$.*

4.2 Theorem. *Let $\phi = \phi_{F,Id}: \mathbb{S}^3 - \mathbf{L} \to \mathbb{S}^1$, defined as above:*

$$\phi(z) = \frac{\overline{i\langle F(z), z\rangle}}{|i\langle F(z), z\rangle|}.$$

Then ϕ is the projection map of a (locally) trivial, C^∞ fibre bundle.

4.3 Theorem. *Each fibre E_θ of ϕ can be compactified by attaching L as its boundary; E_θ is a surface of genus,*

$$g(E_\theta) = \frac{(k-2)(k+1)}{2},$$

and $(k+3)$ boundary components. Moreover, E_θ and $E_{\theta+\pi}$ are glued together along L, forming the closed, smooth surface

$$N_\theta = \{z \in \mathbb{S}^3 \mid \mathrm{Re}\langle e^{i\theta} F(z), z\rangle = 0\},$$

which has genus $g(N_\theta) = k^2$, equal to the (Poincaré-Hopf) local index of the vector field F.

Let us prove Theorem 4.2 first. Let $F = (z_2^k, -z_1^k)$. For each $e^{i\theta} \in \mathbb{S}^1$, let F_θ be the vector field $F_\theta(z) = (e^{i\theta} F(z))$, and define $\psi_\theta(z) =$

$\mathrm{Re}\langle e^{i\theta}F(z), z\rangle$, as before. We claim that if $k > 1$, then $(0,0)$ is the only critical point of ψ_θ. By definition one has,

$$\psi_\theta(z) = \frac{1}{2}\{e^{i\theta}(z_2^k\bar{z}_1 - z_1^k\bar{z}_2) + e^{-i\theta}(\bar{z}_2{}^k z_1 - \bar{z}_1{}^k z_2)\}.$$

Therefore,

$$\frac{\partial\psi_\theta(z)}{\partial z_1} = \frac{1}{2}\{-ke^{i\theta}z_1^{k-1}\bar{z}_2 + e^{-i\theta}\bar{z}_2^k\},$$

and

$$\frac{\partial\psi_\theta(z)}{\partial z_2} = \frac{1}{2}\{ke^{i\theta}z_1^{k-1}\bar{z}_1 - e^{-i\theta}\bar{z}_1^k\}.$$

If $\frac{\partial\psi_\theta(z)}{\partial z_1}$ and $\frac{\partial\psi_\theta(z)}{\partial z_2}$ are both zero then,

$$-ke^{i\theta}z_1^{k-1}\bar{z}_2 + e^{-i\theta}\bar{z}_2^k = 0 = ke^{i\theta}z_2^{k-1}\bar{z}_1 - e^{-i\theta}\bar{z}_1^k.$$

This implies that either $(z_1, z_2) = (0,0)$, or else z_1 and z_2 are both non-zero. In the latter case, dividing the first equation by $\bar{z}_2$ and the second by $\bar{z}_1$, one gets

$$-ke^{i\theta}z_1^{k-1} + e^{-i\theta}\bar{z}_2^{k-1} = 0 = ke^{i\theta}z_2^{k-1} - e^{-i\theta}\bar{z}_1^{k-1},$$

which implies $k = 1$. Hence $(0,0)$ is the only critical point of ψ_θ.

Thus one has:

4.4 Lemma. *Let,*

$$\mathbf{M}_\theta = \{z \in \mathbb{C}^2 \mid e^{i\theta}(z_2^k\bar{z}_1 - z_1^k\bar{z}_2) + e^{-i\theta}(\bar{z}_2{}^k z_1 - \bar{z}_1{}^k z_2) = 0\},$$

with $k > 1$. Then each $\mathbf{M}_\theta - \{0\}$ is a non-empty, smooth, orientable 3-submanifold of $\mathbb{C}^2$.

That $\mathbf{M}_\theta - \{0\} \neq \varnothing$ is a consequence of 1.4 above and 4.5 below.

Let us define a 1-parameter family of diffeomorphisms of $\mathbb{C}^2$ by,

$$h_\alpha(z_1, z_2) = (e^{\frac{-i\alpha}{k-1}}z_1, e^{\frac{-i\alpha}{k-1}}z_2),$$

with $\alpha \in \mathbb{R}$. It is clear that if α is of the form $2\pi r(k-1)$, then h_α is the identity, so the orbits of this flow are all periodic, of period $2\pi(k-1)$. It is clear from the definition that one has $M_\theta = M_{\theta+\pi}$ for all θ. Thus, given $t \in \mathbb{R}$ we identify M_t with $M_{[t]}$, where $[t]$ is t reduced module π.

4.5 Lemma. *Let* $z = (z_1, z_2)$ *and let*

$$(w_1, w_2) = (e^{\frac{-i\alpha}{k-1}} z_1, e^{\frac{-i\alpha}{k-1}} z_2).$$

Then $(z_1, z_2) \in \mathbf{M}_\theta$ *if and only if* $(w_1, w_2) \in \mathbf{M}_{\theta+\alpha}$.

Proof. By definition,

$$\mathbf{M}_{\theta+\alpha} = \{e^{i(\theta+\alpha)}(z_2^k \bar{z}_1 - z_1^k \bar{z}_2) + e^{-i(\theta+\alpha)}(\bar{z}_2{}^k z_1 - \bar{z}_1{}^k z_2) = 0\},$$

and,

$$\mathbf{M}_\theta = \{e^{i\theta}(z_2^k \bar{z}_1 - z_1^k \bar{z}_2) + e^{i\theta}(\bar{z}_2{}^k z_1 - \bar{z}_1{}^k z_2) = 0\}.$$

One has,

$$\begin{aligned} e^{i(\theta+\alpha)}(w_2^k \bar{w}_1 - w_1^k \bar{w}_2) + e^{-i(\theta+\alpha)}(\bar{w}_2{}^k w_1 - \bar{w}_1{}^k w_2) = \\ = e^{i\theta}(z_2^k \bar{z}_1 - z_1^k \bar{z}_2) + e^{-i\theta}(\bar{z}_2{}^k z_1 - \bar{z}_1{}^k z_2). \end{aligned}$$

Hence $(z_1, z_2) \in \mathbf{M}_\theta$ if and only if $(w_1, w_2) \in \mathbf{M}_{\theta+\alpha}$. □

The theorem below summarizes the previous discussion.

4.6 Theorem. *Let* F *be a holomorphic vector field as above. Define a map*

$$\Phi: \mathbb{C}^2 - \{\mathbf{M}\} \to \mathbb{S}^1,$$

by $\Phi(z) = \arg(i\overline{\langle F(z), z\rangle})$. *Then* Φ *is the projection map of a locally trivial fibre bundle over* $\mathbb{S}^1$. *Each fibre* E_θ *is an open* 3*-manifold, that can be "compactified" by attaching the boundary* $\mathbf{M}$. *Furthermore, for each* θ, *the fibers* E_θ *and* $E_{\theta+\pi}$ *are glued together along* $\mathbf{M}$, *forming the real analytic variety* $\mathbf{M}_\theta$ *of points where the vector field* $e^{i\theta}F$ *is tangent to the spheres (as a real vector field);* $\mathbf{M}_\theta - \{0\}$ *is smooth away from* 0.

This theorem essentially implies Theorem 4.2. To complete the proof of 4.2 we observe that the above 1-parameter family of diffeomorphisms preserves the sphere $\mathbb{S}^3$, and the standard action of $\mathbb{R}$ in $\mathbb{C}^2$,

$$t \cdot (z_1, z_2) = (tz_1, tz_2)$$

preserves each $\mathbf{M}_\theta$. Hence each $\mathbf{M}_\theta$ is a cone that intersects transversally the sphere $\mathbb{S}^3$. This proves 4.2. □

The proof of the following lemma is a straight-forward computation. This lemma is also a special case of [**Se**].

4.7 Lemma. *The Jacobian matrix of the map $\psi(z) = \langle F(z), z\rangle$, has rank 2 everywhere except at 0. Hence,* **M** *is smooth away from 0, of dimension 2.*

It follows that **M** is embedded in $\mathbb{C}^2$ as the cone over a knot or link $L \subset \mathbb{S}^3$. We will prove that L consists of $(k+3)$ fibres of the Hopf fibration. For this we recall how the Hopf fibration looks like. One has two special fibres, given by the intersection of $\mathbb{S}^3$ with the two axis, which define the Hopf link. If we remove from $\mathbb{S}^3$ these two fibres we get a thickened open torus $\mathbb{T}^2 \times (-1,1)$. This can be foliated by tori $\mathbb{T}^2 \times t$, and each such torus can be foliated by torus knots of type $(1,1)$. These are the fibres of the Hopf fibration.

Let us prove Theorem 4.1. By definition one has that **M** is the set of points (z_1, z_2) that satisfy,

$$z_2^k \bar{z}_1 = z_1^k \bar{z}_2.$$

Hence one has,

4.8 Lemma. **M** *is the union of the two axes $\{z_1 = 0\}$ and $\{z_2 = 0\}$, with the real analytic variety,*

$$\mathbf{M}^- = \{z_2^{k-1}(\frac{z_2}{\bar{z}_2}) = z_1^{k-1}(\frac{z_1}{\bar{z}_1}); z_1 \neq 0 \neq z_2\}.$$

This means that every point in $\mathbf{M}^-$ satisfies:

i) $\|z_1\| = \|z_2\|$, and

ii) $(k+1)\arg(z_1) = (k+1)\arg(z_2)$.

Hence one has,

4.9 Lemma. $\mathbf{M}^-$ *can be parametrized by,*

$$\mathbf{M}^- = \{(z_1, z_2) \mid z_1 = \mathrm{r}e^{i\theta} \quad \textit{and} \quad z_2 = \mathrm{r}e^{i(\theta + \frac{2s\pi}{k+1})}\},$$

with $s = 0, 1, \ldots, k$, $r \geq 0$ and $\theta \in [0, 2\pi)$.

Proof. We already know that $\|z_1\| = \|z_2\| = r$, for some $r \geq 0$. To prove that if $(z_1, z_2) \in \mathbf{M}^-$ then their arguments are as stated in the lemma (and vice versa), is a straightforward computation.

If $(z_1, z_2) \in \mathbf{M} \cap \mathbb{S}^3$, then $r = \frac{\sqrt{2}}{2}$; for each fixed θ one has 1 point in the z_1-axis, the point $\frac{\sqrt{2}}{2} \cdot e^{i\theta}$, and $(k+1)$ points in the z_2-axis, the points

$$\frac{\sqrt{2}}{2} \cdot e^{i(\theta + \frac{2s\pi}{k+1})}, s = 0, \ldots, k.$$

As the angle increases from θ to $0 + 2\pi$, we go round the first axis once, and each of the $(k+1)$-points in the z_2-axis also goes round once. Therefore $\mathbf{M}^- \cap \mathbb{S}^3$ consists of $(k+1)$ torus knots of type $(1, 1)$. These are $(k+1)$ fibres of the Hopf fibration, which together with two fibres of the Hopf link, shows that L consists of $(k+3)$ fibres of the Hopf fibration, proving Theorem 4.1.

Let us now determine the monodromy map of the fibre bundle,

$$\begin{aligned} \phi\colon \mathbb{S}_\epsilon - L &\to \mathbb{S}^1, \\ z &\mapsto \frac{\overline{i\langle F(z), z\rangle}}{|i\langle F(z), z\rangle|}. \end{aligned}$$

We already know that the above flow $\{h_\alpha\}$, transports fibres onto fibres. Moreover, the orbits of this flow can be regarded as liftings of the generator of π_1 of the base $\mathbb{S}^1$. Thus, the monodromy h is the first return map of this flow. To determine h, take a point (z_1, z_2) in some fibre E_θ, then $h(z_1, z_2)$ is the first point in E_θ of the form,

$$(w_1, w_2) = (e^{\frac{-i\theta}{k-1}} z_1, e^{\frac{-i\theta}{k-1}} z_2),$$

for some $\alpha > 0$. But we know that (w_1, w_2) is in $M_{\theta+\alpha}$, so that α must be a multiple of 2π. Hence

$$h(z_1, z_2) = (e^{\frac{-2\pi i}{k-1}} z_1, e^{\frac{-2\pi i}{k-1}} z_2),$$

so it is periodic, of period $(k-1)$.

We now recall that $\mathbb{S}^3 - \mathbf{L}$ is the total space of the Hopf fibration minus $(k+3)$ fibres. This means $\mathbb{S}^3 - \mathbf{L}$ is $\mathbb{S}^1 \times (\mathbb{S}^2 - (k+3)\text{-points})$. Hence the monodromy h maps E_θ onto $(\mathbb{S}^2 - (k+3)\text{-points})$ as a $(k-1)$-fold cover. Thus, if we attach a 2-disc $\mathbb{D}^2$ for each boundary component of E_θ, we get a closed surface $\hat{E}_\theta$ which is a branched cover of $\mathbb{S}^2$, ramified at $(k+3)$ points. Hence, by Hurwitz formula, the Euler-Poincaré characteristic of $\hat{E}_\theta$ is,

$$\chi(\hat{E}_\theta) = (k+3) - (k-1)(k+2) + (k-1) = -k^2 + k + 4.$$

Thus E_θ has genus,

$$g(E_\theta) = \frac{(k-2)(k+1)}{2},$$

as stated in Theorem 4.3.

By Theorem 1.4, the fibres E_θ and $E_{\theta+\pi}$ are glued together along $\mathbf{L}$ forming the closed smooth surface,

$$N_\theta = \{\mathrm{Re}\langle e^{i\theta}F(z), z\rangle = 0\} \cap \mathbb{S}^3.$$

Topologically, N_θ is the double of E_θ, so its genus is twice the genus of E_θ plus the number of new handles that we create when we identify the boundaries. Since we have $(k+3)$-boundary components on each fibre, we increase the genus by $(k+2)$. Hence the genus of N_θ is,

$$g(N_\theta) = (k-2)(k+1) + k + 2 = k^2,$$

thus we arrive to Theorem 4.3. [That k^2 is the Poincaré-Hopf index of F follows from the well known fact, that the local index of F equals its algebraic multiplicity

$$\mu = \dim \frac{\mathcal{O}_{2}, 0}{(z_1^k, z_2^k)},$$

which is k^2.] □

The same approach can be used in other cases. For instance, if $F = (z_1^k, z_2^k)$, $k > 1$, one gets that M is the cone over $(k+1)$ fibers of the Hopf fibration, the corresponding vector bundle is also trivial, and every two antipodal fibers are glued together forming a closed surface in the 3-sphere, whose genus is $(k-1)^2$.

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José A. Seade
Instituto de Matemáticas,
Universidad Nacional Autónoma de México
Ciudad Universitaria,
México 04510 D.F., México
e-mail: jseade@ gauss.matem.unam.mx